# Numerical solutions of boundary value problems.

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# A mechanical pendulum

Nonlinear problem

$$m\frac{d^2\varphi}{dt^2} + \omega^2 \sin\varphi = 0,$$

where $\omega^2 = \frac{g}{h}$ is for small angles $\varphi$ linearized as

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$$\varphi(t = t_0) = \varphi_0, \quad \frac{d\varphi}{dt}(t = t_0) = \varphi_1$$

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# A heat equation - 1D rod model

$$\frac{du}{dt} - \alpha \frac{d^2u}{dx^2} = \frac{1}{c_p \rho} q, \quad \alpha = \frac{k}{c_p \rho}$$

Parameters:

- $c_p$ - specific heat capacity
- $k$ - thermal conductivity
- $\rho$ - density
- $q$ - internal heat (source term)

Special cases: No internal heat + stationary case:

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Discussion: 1) boundary conditions? 2) Extension to 3D?

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# A linear elasticity in 1D/2D/3D

Navier - Lamme equations using Einstein notation:

$$(\lambda+\mu)u_{k,ki}+\mu\, u_{i,kk}+F_i=\rho\frac{d^2u_i}{dt^2}$$

Equivalently:

$$(\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u}+\mu\nabla^2\mathbf{u}+\mathbf{F_i}=\rho\frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}t^2}$$

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# Example of FEM: coarse mesh

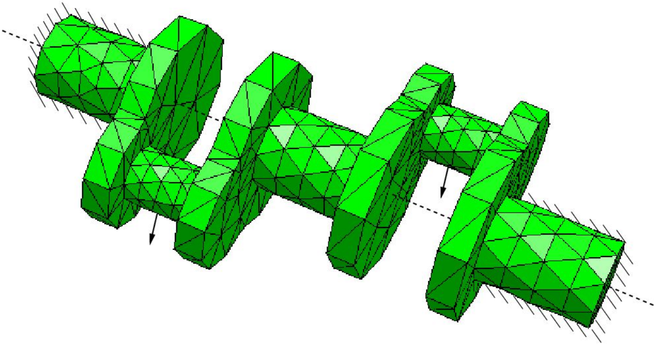

**Figure:** The coarse mesh of the crankshaft model.

# Example of FEM: elastoplastic field

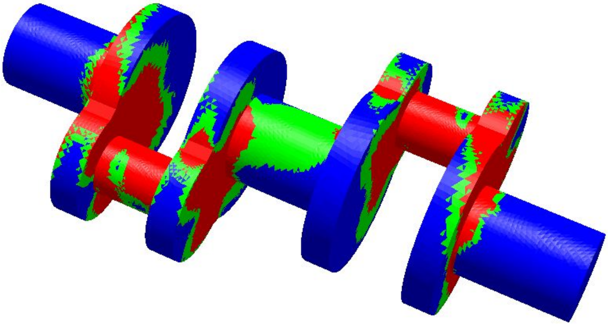

**Figure:** Fine mesh with 808448 tetrahedra obtained by three uniform refinements of the coarse mesh.

# Finite elements methods

FEM is a numerical technique for finding approximate solutions to boundary value problems, such as

- diffusion equations, heat condution
- elasticity, nonliner elasticity (plasticity)
- equations of flows in fluids (Navier-Stokes equations)
- electro-magnetic equations (Maxwell)
- ...

# History of FEM

- works of Alexander Hrennikoff (1941) and Richard Courant (1942)
- John Argyris (Stuttgart)
- Ray W. Clough (Berkeley)
- software NASTRAN by NASA at the end of 1960s.
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# FEM for 1D Poisson problem: strong and weak formulations

We solve

$$-u''(x) = f(x) \quad x \in (0,1)$$
$$u(0) = u(1) = 0,$$

equivalent to the weak formulation

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# FEM for 1D Poisson problem: linear basis functions

We assumme the uniform triangulation with points:

$$x_0 = 0, \quad x_1 = h, \quad x_2 = 2h, \quad \dots, \quad x_{N+1} = 1,$$

where $h = 1/N + 1$ and look for the solution in the form

$$u = \sum_{i=0}^{N+1} u_i \varphi_i,$$

where $\varphi_i$, $i = 0 \dots N + 1$ represents a set of finite element basis functions.

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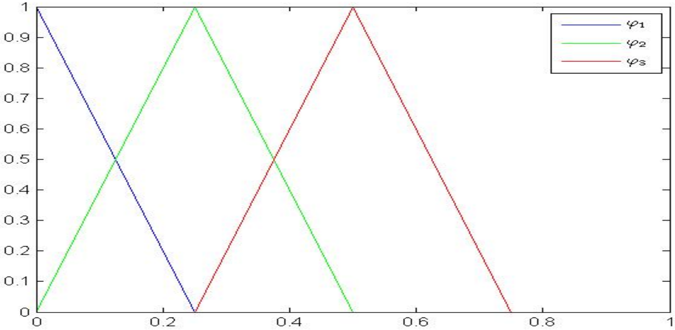


**Figure:** Nodal basis functions

# FEM for 1D Poisson problem: linear systems of equations

The substitutions of $u$ in the weak form results in the linear systems of equations

$$Au = b,$$

where

$$A_{ij} = \int_0^1 \varphi_i' \varphi_j' dx, \quad b_i = \int_0^1 f \varphi_i dx$$

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# Examples

- Poisson problem in 1D
- Obstacle problem in 2D

Some MATLAB examples are available at Matlab Central under `http://www.mathworks.com/matlabcentral/fileexchange/authors/37756`.

Thank You for Your Attention.